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Development of a Modified Probability Distribution, Properties and its Applications to Biomedical Data sets

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ABSTRACT
This paper introduces and explores an expanded version of the conventional Gumbel type II distribution, referred to as the Weibull-Gumbel type II distribution. Characterized by a singular scale parameter and three shape parameters, this innovative lifetime distribution is subjected to a comprehensive analysis, employing the maximum product of spacing method for parameter estimation. The study utilizes two real-life datasets to showcase the adaptability and versatility of the Weibull-Gumbel type II distribution. Assessment based on log-likelihood and information statistics values from both estimation methods reveals that this distribution offers a superior fit to the data when compared to alternative distributions. Furthermore, a simulation study confirms the consistency of the parameters. Given these results, the Weibull-Gumbel type II distribution is recommended as an effective model for the accurate representation of lifetime data

KEYWORDS
Weibul-Gumbel type II, maximum product of spacing, monotonic decreasing and increasing shapes, Monte Carlo simulation, binomial expansion

1. Introduction

Numerous statistical distributions have been recently devised and investigated in scholarly literature. Despite the emergence of these new distributions, there remains a need for models that align with current circumstances, underscoring the continuous exploration in this domain. By incorporating flexibility into basic probability distributions to accommodate unique real-world events, the potential for broader applications of these new models is heightened. This inclination has motivated researchers to concentrate on the development of novel, adaptable distributions. Various methods to extend standard probability distributions have been proposed in the literature. One prevalent approach involves the use of distribution generators, exemplified by the exponentiated family of distributions introduced by Nadarajah and Kotz [10], the Kumaraswamy generalized family of distributions by Cordeiro and De Castro [6], and the Weibull-G family of distributions explored by Bourguignon et al. [5]. Additional contributions

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include the Topp-Leone generalized family of distributions by Al-Shomrani et al. [1], the exponentiated extended generalized family of distributions by Elgarhy et al. [7], the Power Lindley generalized family of distributions by Hassan and Nassr [9], and others, as enumerated in the subsequent citations. In the context of extreme values, such as the maximum or minimum of a large set of independently distributed random variables, the Gumbel type II (GuTII) distribution, also known as the extreme value type II distribution, is commonly employed. Its probability density function (pdf) and cumulative distribution function (cdf) serve to define it as follows:

$$G(x) = e^{-\theta x^{-\sigma}} \tag{1}$$

$$g(x) = \sigma\theta x^{-\sigma-1} e^{-\theta x^{-\sigma}} \tag{2}$$

The Gumbel type II (GuTII) distribution stands as a valuable model within extreme value theory, finding applications in fields such as seismology and meteorology where it is employed to simulate extreme events. "This distribution proves beneficial in risk management, operational risk, and life testing, particularly when characterizing lifetime datasets exhibiting monotonic failure rates. However, a notable limitation arises when dealing with the majority of complex events observed in practical scenarios, as they often exhibit non-monotonic behavior. To address this limitation, the GuTII distribution is combined with the Weibull-G family of distributions to enhance flexibility and improve fitting. To enhance the fit of the GuTII distribution, Okorie et al. [12] proposed and investigated an exponentiated form of the GuTII distribution of Lehman type I. Ogunde et al. [11] extended the GuTII distribution using a generalized exponentiated-G distribution, leading to Lehman type I and type II GuTII distributions. Additionally, Okorie et al. [13] explored the characteristics of the Kumaraswamy-G Exponentiated GuTII distribution. In the realm of statistical representation, the Weibull distribution has been widely employed over the past few decades in fields such as dependability, engineering, and biological studies. Gurvich et al. [8] expanded the traditional Weibull model to encompass a broader family of univariate distributions. Bourguignon et al. [5] delved into the exploration of this expanded family, termed the Weibull-G family of distributions, providing insights into its mathematical characteristics. This approach, involving the addition of extra shape parameters to the standard distribution, offers a useful means of enhancing resilience and flexibility. The cumulative distribution function (cdf) and probability density function (pdf) of the Weibull-G family of distributions are provided as part of this exploration.

$$F(x) = 1 - e^{-\alpha\left[\frac{G(x)}{1-G(x)}\right]^{\beta}} \tag{3}$$

$$f(x) = \alpha\beta g(x)\frac{[G(x)]^{\beta-1}}{[1-G(x)]^{\beta+1}} e^{-\alpha\left[\frac{G(x)}{1-G(x)}\right]^{\beta}} \tag{4}$$

This work aims to generalize the classical GuTII distribution to a wider class of distribution in order to improve its performance and fit, and also promote its usefulness in modeling various complicated data sets.

2. Weibull-Gumbel Type II (WGuTII) distribution

In this segment, a novel continuous probability density function (pdf) known as the Weibull-Gumbel Type II distribution is introduced. Plots depicting the pdf, cumulative distribution function (cdf), survival function, and hazard rate function (hrf) are presented to evaluate the characteristics of this newly developed distribution. The cumulative distribution function (cdf) for the Weibull-Gumbel Type II (WGuTII) distribution is derived by substituting equation (1) into equation (3), resulting in:

$$F(x) = 1 - e^{-\alpha\left[\frac{e^{-\theta x^{-\sigma}}}{1-e^{-\theta x^{-\sigma}}}\right]^{\beta}} \tag{5}$$

Also, the pdf of WGuII distribution is obtained by inserting (2) into (4) as:

$$f(x) = \alpha\beta\sigma\theta x^{-\sigma-1} e^{-\theta x^{-\sigma}} \frac{\left[e^{-\theta x^{-\sigma}}\right]^{\beta-1}}{[1 - e^{-\theta x^{-\sigma}}]^{\beta+1}} e^{-\alpha\left[\frac{e^{-\theta x^{-\sigma}}}{1-e^{-\theta x^{-\sigma}}}\right]^{\beta}} \tag{6}$$

where $x \geq 0$, α, $\theta > 0$ are the scale parameters and $\alpha, \beta > 0$ are the shape parameters respectively.

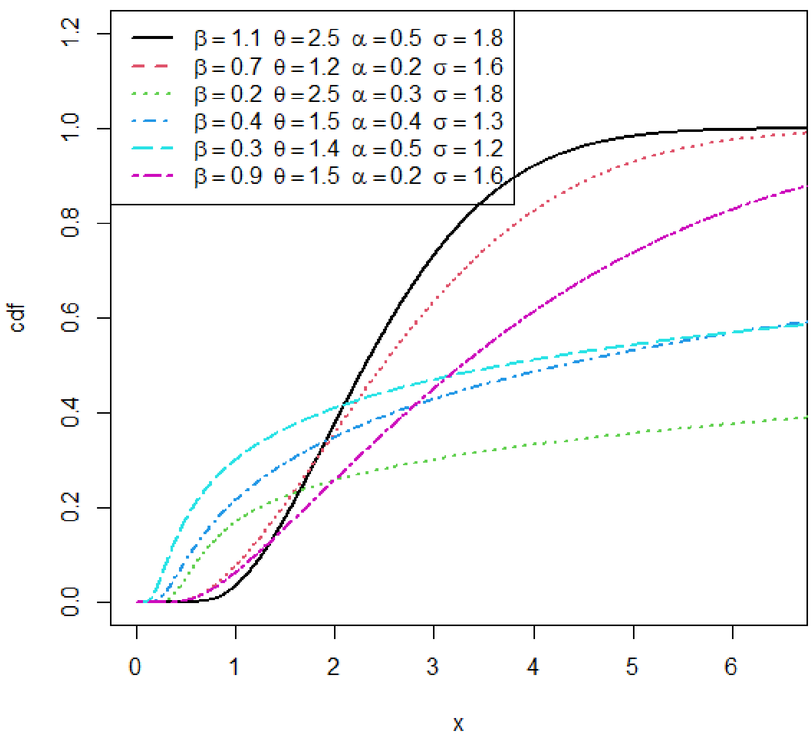

Figure 1. cdf plots of WGuTII distribution with different parameter values

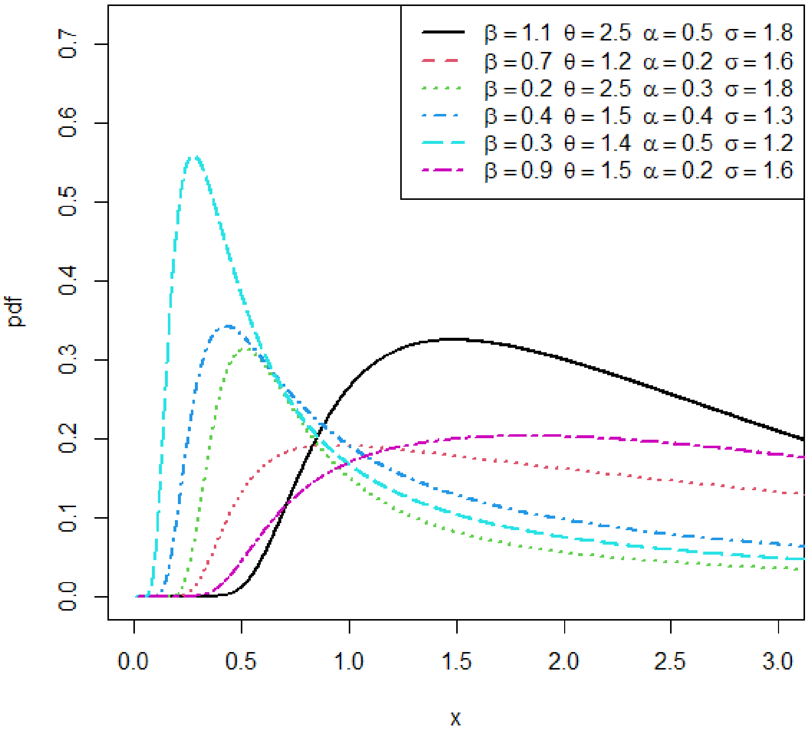

Figure 2. pdf plots of WGuTII distribution with different parameter values

3. Density Expansion

In this section the pdf in (6) is expanded using binomial expansion. This is obtained a follows:

$$f(x) = \alpha\beta\sigma\theta x^{-\sigma-1}e^{-\theta x^{-\sigma}}\frac{\left[e^{-\theta x^{-\sigma}}\right]^{\beta-1}}{[1-e^{-\theta x^{-\sigma}}]^{\beta+1}}$$

$$e^{-\alpha\left[\frac{e^{-\theta x^{-\sigma}}}{1-e^{-\theta x^{-\sigma}}}\right]^{\beta}} = \sum_{i=0}^{\infty}(-1)^i(\alpha)^i\left[\frac{e^{-\theta x^{-\sigma}}}{1-e^{-\theta x^{-\sigma}}}\right]^{\beta i} \tag{7}$$

$$\left[1-e^{-\theta x^{-\sigma}}\right]^{-(\beta(1+i)+1)} = \sum_{j=0}^{\infty}(-1)^j\begin{pmatrix} -(\beta(1+i)+1) \\ j \end{pmatrix}[e^{-\theta x^{-\sigma}}]^j \tag{8}$$

On combining (7) and (8) together then substituting back to (6), we have

$$f(x) = \alpha\beta\sigma\theta x^{-\sigma-1} \sum_{i,j=0}^{\infty} (-1)^{i+j} (-\alpha)^i \left(\begin{array}{c} -(\beta(1+i)+1) \\ j \end{array} \right) [e^{-\theta x^{-\sigma}}]^{\beta(i+1)+j} \tag{9}$$

In the same vain, (5) is expanded as follows

$$[F(x)]^h = \left[1 - e^{-\alpha} \left[\frac{e^{-\theta x^{-\sigma}}}{1 - e^{-\theta x^{-\sigma}}} \right]^{\beta} \right]^h \tag{10}$$

$$\left[1 - e^{-\alpha} \left[\frac{e^{-\theta x^{-\sigma}}}{1 - e^{-\theta x^{-\sigma}}} \right]^{\beta} \right]^h = \sum_{k=0}^{h} (-1)^k \left(\begin{array}{c} h \\ k \end{array} \right) \left[e^{-\alpha \left[\frac{e^{-\theta x^{-\sigma}}}{1 - e^{-\theta x^{-\sigma}}} \right]^{\beta}} \right]^k \tag{11}$$

$$e^{-\alpha k \left[\frac{e^{-\theta x^{-\sigma}}}{1 - e^{-\theta x^{-\sigma}}} \right]^{\beta}} = \sum_{p=0}^{\infty} (-1)^p (\alpha k)^p \left[\frac{e^{-\theta x^{-\sigma}}}{1 - e^{-\theta x^{-\sigma}}} \right]^{\beta p} \tag{12}$$

$$\left[1 - e^{-\theta x^{-\sigma}} \right]^{-\beta k} = \sum_{q=0}^{\infty} (-1)^q \left(\begin{array}{c} -\beta k \\ q \end{array} \right) \left[e^{-\theta x^{-\sigma}} \right]^q \tag{13}$$

On combining (11),(12) and (13) then substituting back to (10), we have

$$[f(x)]^h = \sum_{p,q=0}^{\infty} \sum_{k=0}^{h} (-1)^{k+p+q} (-\alpha k)^p \left(\begin{array}{c} -\beta k \\ q \end{array} \right) \left(\begin{array}{c} h \\ k \end{array} \right) \left[e^{-\theta x^{-\sigma}} \right]^q \tag{14}$$

4. Properties of WGuTII distribution

In this segment, our examination focuses on the statistical properties of the WGuTII distribution, with a specific emphasis on the survival function, hazard function, quantile function, moments, and moment-generating function.

4.1. *Reliability (Survival) function*

$$R(x) = e^{-\alpha k \left[\frac{e^{-\theta x^{-\sigma}}}{1 - e^{-\theta x^{-\sigma}}} \right]^{\beta}} \tag{15}$$

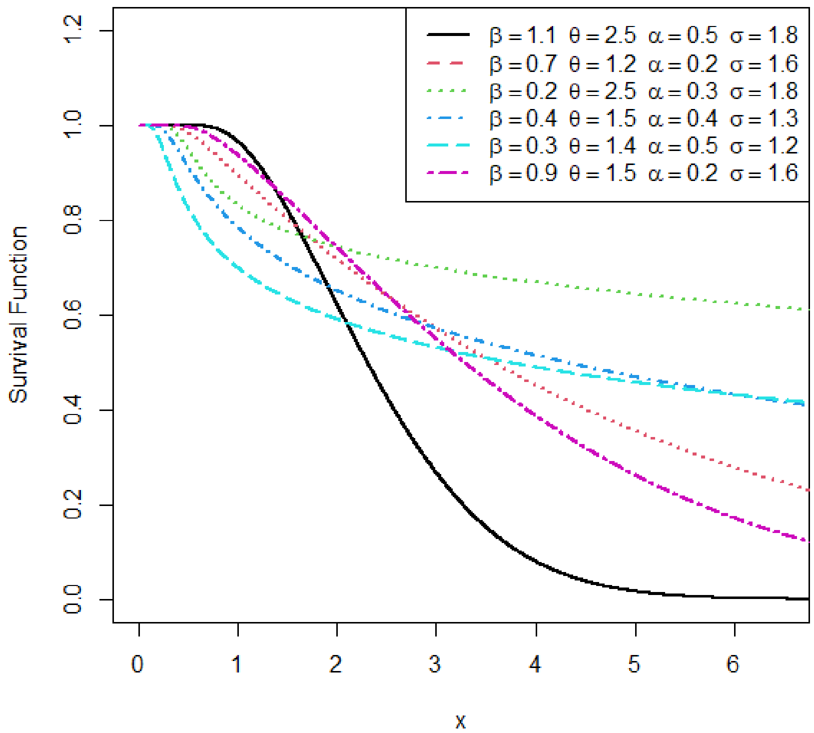

Figure 3. Survival function plots of WGuTII distribution with different parameter values

4.2. *Hazard Function*

$$H(x) = \alpha\beta\sigma\theta x^{-\sigma-1} e^{-\theta x^{-\sigma}} \frac{\left[e^{-\theta x^{-\sigma}}\right]^{\beta-1}}{[1-e^{-\theta x^{-\sigma}}]^{\beta+1}} \tag{16}$$

4.3. *Quantile Function*

Quantile function has a significant position in probability theory and it is the inverse of the cdf. The quantile function is obtained using

$$Q(u) = F^{-1}(u)$$

using the inverse of equation (5), we have the quantile function give as

$$x = Q(u) = \left\{ \frac{-1}{\theta} \log \left[\frac{\log(1-u)}{\log(1-u) - \alpha\beta} \right]^{\frac{-1}{\sigma}} \right\} \tag{17}$$

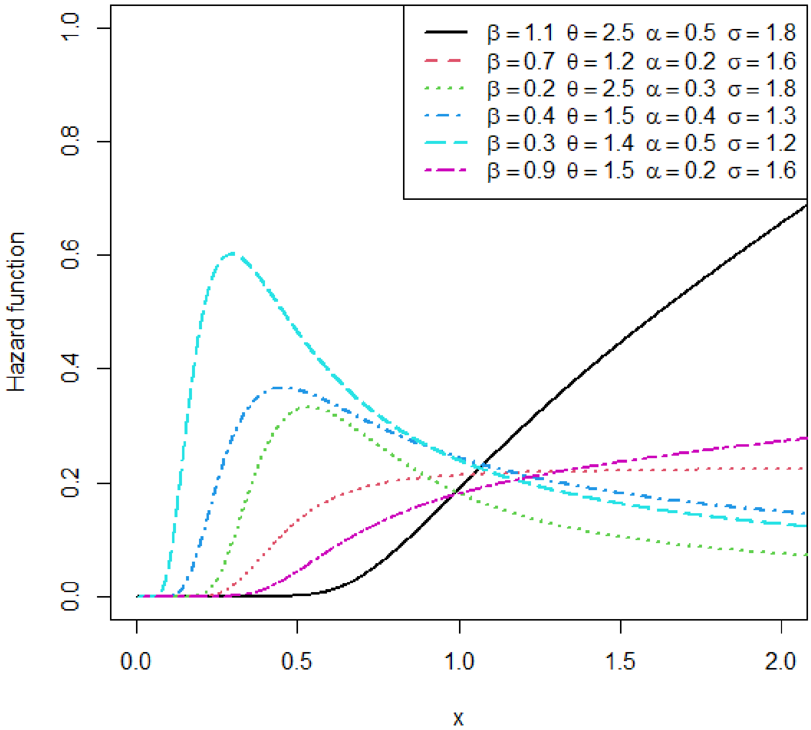

Figure 4. Hazard function plots of WGuTII distribution with different parameter values

4.4. *Moment*

Here, we consider the rth moment for WGuTII distribution. Moments are important features in any statistical analysis, especially in applications. They can be used to study the characteristics of a distribution, e.g., dispersion, skewness, and kurtosis.

$$E(X^r) = \int_0^\infty x^r f(x)dx \tag{18}$$

$$E(X^r) = \alpha\beta\sigma\theta x^{-\sigma-1} \sum_{i,j=0}^{\infty} (-1)^{i+j} (-\alpha)^i \binom{-(\beta(1+i)+1)}{j} \int_0^\infty x^r [e^{-\theta x^{-\sigma}}]^{\beta(i+1)+j} dx \tag{19}$$

$$E(X^r) = \int_0^\infty x^r [e^{-\theta x^{-\sigma}}]^{\beta(i+1)+j} dx \tag{20}$$

Let

$$(\beta(i+1)+j)\theta x^{-\delta} \implies x = \left[\frac{y}{(\beta(i+1)+j)\theta}\right]^{\frac{-1}{\delta}} dx = \frac{dy}{\delta(\beta(i+1)+j)\theta x^{-\delta-1}}$$

$$\int_0^\infty \left[\frac{y}{(\beta(i+1)+j)\theta}\right]^{\frac{-1}{\delta}} e^{-y} \frac{dy}{\delta(\beta(i+1)+j)\theta x^{-\delta-1}} = \int_0^\infty y^{\frac{-r}{\delta}} e^{-y} dy$$

$$\int_0^\infty y^{\frac{-r}{\delta}} e^{-y} dy = \Gamma\left(1-\frac{r}{\delta}\right)$$

$$E(X^r) = \alpha\beta\theta^{\frac{r}{\sigma}}[\beta(i+1)+j]^{\frac{r}{\sigma}+1} \sum_{i,j=0}^{\infty} (-1)^{i+j}(-\alpha)^i \binom{-(\beta(1+i)+1)}{j} \Gamma\left(1-\frac{r}{\sigma}\right) \tag{21}$$

The mean of WGuTII distribution is obtained by setting $r = 1$ in (21)

$$E(X) = \alpha\beta\theta^{\frac{1}{\sigma}}[\beta(i+1)+j]^{\frac{1}{\sigma}+1} \sum_{i,j=0}^{\infty} (-1)^{i+j}(-\alpha)^i \binom{-(\beta(1+i)+1)}{j} \Gamma\left(1-\frac{1}{\sigma}\right) \tag{22}$$

4.5. *Moment generating function*

$$M_{(x)}(t) = \int_0^\infty e^{ix} f(x)dx \tag{23}$$

since the series expansion for e^{ix} is given as

$$e^{ix} = \sum_{w=0}^{\infty} \frac{(tx)^w}{w!} \tag{24}$$

$$M_{(x)}(t) = \alpha\beta\theta^{\frac{w}{\sigma}}[\beta(i+1)+j]^{\frac{w}{\sigma}+1} \sum_{w=0}^{\infty} \frac{(tx)^w}{w!} \sum_{i,j=0}^{\infty} (-1)^{i+j}(-\alpha)^i \binom{-(\beta(1+i)+1)}{j} \Gamma\left(1-\frac{w}{\sigma}\right) \tag{25}$$

4.6. *Order Statistics*

Let $X_1, X_2, \ldots, X_n$ be n independent random variable from the WGuTII distributions and let $X_{(1)} \leq X_{(2)} \leq \ldots \leq X_{(n)}$ be their corresponding order statistic. Let $F_{r,n}(x)$ and $f_{r,n}(x)$, $r = 1, 2, 3, \ldots, n$ denote the cdf and pdf of the rth order statistics $X_{r,n}(x)$ respectively. The pdf of the rth order statistics of $X_{r,n}(x)$ is given as

$$f_{r,n}(x) = \frac{f(x)}{B(r, n-r+1)} \sum_{v=0}^{n-r} (-1)^v \binom{n-r}{v} F(x)^{v+r-1} \tag{26}$$

The pdf of rth order statistic for distribution is obtained also replacing h with $v+r-1$ in cdf expansion. we have

$$\begin{aligned} f_{r,n}(x) &= n\alpha\beta\sigma\theta x^{-\sigma-1}\frac{1}{B(r,n-r+1)}\sum_{i,j,p,q=0}^{\infty}\sum_{v=0}^{n-r}\sum_{k=0}^{v+r-1}(-1)^{v+i+j+k+p+q} \\ & (-\alpha)^i(-\alpha k)^p\begin{pmatrix} (-\beta(i+1)+1) \\ j \end{pmatrix}\begin{pmatrix} n-r \\ v \end{pmatrix}\begin{pmatrix} -\beta k \\ q \end{pmatrix}\begin{pmatrix} v+r-1 \\ k \end{pmatrix} \\ & \left[e^{-\theta x^{-\sigma}}\right]^{\beta(i+1)+j+q} \end{aligned} \quad (27)$$

The pdf of minimum order statistic of the distribution is obtained by setting $r = 1$

$$\begin{aligned} f_{1,n}(x) &= n\alpha\beta\sigma\theta x^{-\sigma-1}\sum_{i,j,p,q=0}^{\infty}\sum_{v=0}^{n-1}\sum_{k=0}^{v}(-1)^{v+i+j+k+p+q}(-\alpha)^i(-\alpha k)^p \\ & \begin{pmatrix} (-\beta(i+1)+1) \\ j \end{pmatrix}\begin{pmatrix} n-1 \\ v \end{pmatrix}\begin{pmatrix} -\beta k \\ q \end{pmatrix}\begin{pmatrix} v \\ k \end{pmatrix} \\ & \left[e^{-\theta x^{-\sigma}}\right]^{\beta(i+1)+j+q} \end{aligned} \quad (28)$$

Also, the pdf of maximum order statistic of the distribution is obtained by setting $r = n$

$$\begin{aligned} f_{n,n}(x) &= n\alpha\beta\sigma\theta x^{-\sigma-1}\sum_{i,j,p,q=0}^{\infty}\sum_{k=0}^{v+n-1}(-1)^{v+i+j+k+p+q}(-\alpha)^i(-\alpha k)^p \\ & \begin{pmatrix} (-\beta(i+1)+1) \\ j \end{pmatrix}\begin{pmatrix} -\beta k \\ q \end{pmatrix}\begin{pmatrix} v+n-1 \\ k \end{pmatrix} \\ & \left[e^{-\theta x^{-\sigma}}\right]^{\beta(i+1)+j+q} \end{aligned} \quad (29)$$

5. Parameter Estimation

This section provides estimation method to estimate the unknown parameters of WGuTII distribution. The methods used to estimate the parameters is maximum product spacing estimation methods.

5.1. *Maximum Product Spacing Estimation (MPS)*

Let $x_1, x_2, \ldots, x_n$ be a random samples from the WGuTII distribution having cdf $F(x; \sigma, \alpha, \theta, \beta)$ and $x_1, x_2, \ldots, x_n$ represents the corresponding ordered sample. The spacing

$$\Psi_i = F(x_{(i)}) - F(x_{(i-1)}) \text{ for } i = 1, 2, \ldots, n+1$$

where $F(x_{(0)}) = 0$ and $F(x_{(n+1)}) = 1$. Therefore,

$$F(x_{(i)}; \sigma, \alpha, \theta, \beta) = 1 - e^{-\alpha\left[\frac{e^{-\theta x_{(i)}^{-\alpha}}}{1-e^{-\theta x_{(i)}^{-\alpha}}}\right]^{\beta}} \tag{30}$$

and

$$F(x_{(i-1)}; \sigma, \alpha, \theta, \beta) = 1 - e^{-\alpha\left[\frac{e^{-\theta x_{(i-1)}^{-\alpha}}}{1-e^{-\theta x_{(i-1)}^{-\alpha}}}\right]^{\beta}} \tag{31}$$

Thus,

$$\Psi_i = \left\{\left[1 - e^{-\alpha\left[\frac{e^{-\theta x_{(i)}^{-\alpha}}}{1-e^{-\theta x_{(i)}^{-\alpha}}}\right]^{\beta}}\right] - \left[1 - e^{-\alpha\left[\frac{e^{-\theta x_{(i-1)}^{-\alpha}}}{1-e^{-\theta x_{(i-1)}^{-\alpha}}}\right]^{\beta}}\right]\right\} \tag{32}$$

The parameter estimates are obtained by maximizing

$$\Omega(x; \sigma, \alpha, \theta, \beta) = \frac{1}{n+1}\sum_{i=1}^{n} \log \Psi_i$$

$$\Omega(x; \sigma, \alpha, \theta, \beta) = \frac{1}{n+1}\sum_{i=1}^{n} \log\left\{\left[1 - e^{-\alpha\left[\frac{e^{-\theta x_{(i)}^{-\alpha}}}{1-e^{-\theta x_{(i)}^{-\alpha}}}\right]^{\beta}}\right] - \left[1 - e^{-\alpha\left[\frac{e^{-\theta x_{(i-1)}^{-\alpha}}}{1-e^{-\theta x_{(i-1)}^{-\alpha}}}\right]^{\beta}}\right]\right\} \tag{33}$$

Differentiating Ω with respect to individual parameters yields the parameter estimates of $\hat{\sigma}_{MPS}, \hat{\alpha}_{MPS}, \hat{\theta}_{MPS}, \hat{\beta}_{MPS}$ and solving the non-linear equations, we have

$$\frac{\partial\Omega(x; \sigma, \alpha, \theta, \beta)}{\partial\sigma} = \frac{1}{n+1}\sum_{i=1}^{n+1} \frac{1}{\Psi_i}[K_1(x_{(i)}; \sigma, \alpha, \theta, \beta) - K_2(x_{(i-1)}; \sigma, \alpha, \theta, \beta)] \tag{34}$$

where

$$K_1(x_{(i)}; \sigma, \alpha, \theta, \beta) = \frac{e^{-\alpha\left[\frac{e^{-\theta x_{(i)}^{-\alpha}}}{1-e^{-\theta x_{(i)}^{-\alpha}}}\right]^{\beta}} e^{-\alpha\left[\frac{e^{-\theta x_{(i)}^{-\alpha}}}{1-e^{-\theta x_{(i)}^{-\alpha}}}\right]^{\beta-1}} \theta e^{-\theta x_{(i)}^{-\sigma}} x_{(i)} \ln x_{(x_i)} \left[\left[1 - e^{-\theta x_{(i)}^{-\sigma}}\right] + e^{-\theta} x_{(i)}^{-\sigma}\right]}{1 - e^{-\alpha\left[\frac{e^{-\theta x_{(i)}^{-\alpha}}}{1-e^{-\theta x_{(i)}^{-\alpha}}}\right]^{\beta}} \left[1 - e^{-\theta x_{(i)}^{-\alpha}}\right]^2}$$

and

$$K_2(x_{(i-1)};\sigma,\alpha,\theta,\beta)=\frac{e^{-\alpha\left[\frac{e^{-\theta x_{(i-1)}^{-\alpha}}}{1-e^{-\theta x_{(i-1)}^{-\alpha}}}\right]^{\beta}} e^{-\alpha\left[\frac{e^{-\theta x_{(i-1)}^{-\alpha}}}{1-e^{-\theta x_{(i-1)}^{-\alpha}}}\right]^{\beta-1}} \theta e^{-\theta x_{(i-1)}^{-\sigma}} x_{(i-1)} \ln x_{(x_i)} \left[\left[1-e^{-\theta x_{(i-1)}^{-\sigma}}\right]+e\right.}{\left(1-e^{-\alpha\left[\frac{e^{-\theta x_{(i-1)}^{-\alpha}}}{1-e^{-\theta x_{(i-1)}^{-\alpha}}}\right]^{\beta}}\right)\left[1-e^{-\theta x_{(i-1)}^{-\alpha}}\right]^2}$$

$$\frac{\partial\Omega(x;\sigma,\alpha,\theta,\beta)}{\partial\theta}=\frac{1}{n+1}\sum_{i=1}^{n+1}\frac{1}{\Psi_i}[P_1(x_{(i)};\sigma,\alpha,\theta,\beta)-P_2(x_{(i-1)};\sigma,\alpha,\theta,\beta)] \qquad (35)$$

where

$$P_1(x_{(i)};\sigma,\alpha,\theta,\beta)=\frac{e^{-\alpha\left[\frac{e^{-\theta x_{(i)}^{-\sigma}}}{1-e^{-\theta x_{(i)}^{-\sigma}}}\right]^{\beta}} e^{-\sigma\left[\frac{e^{-\theta x_{(i)}^{-\sigma}}}{1-e^{-\theta x_{(i)}^{-\sigma}}}\right]^{\beta-1}} \theta e^{-\theta x_{(i)}^{-\sigma}} x_{(i)} \ln x_{(x_i)} \left[\left[1-e^{-\theta x_{(i)}^{-\sigma}}\right]+e^{-\theta} x_{(i)}^{-\sigma}\right]}{\left(1-e^{-\alpha\left[\frac{e^{-\theta x_{(i)}^{-\sigma}}}{1-e^{-\theta x_{(i)}^{-\sigma}}}\right]^{\beta}}\right)\left[1-e^{-\theta x_{(i)}^{-\sigma}}\right]^2}$$

and

$$P_2(x_{(i-1)};\sigma,\alpha,\theta,\beta)=\frac{e^{-\alpha\left[\frac{e^{-\theta x_{(i-1)}^{-\sigma}}}{1-e^{-\theta x_{(i-1)}^{-\sigma}}}\right]^{\beta}} e^{-\alpha\left[\frac{e^{-\theta x_{(i-1)}^{-\sigma}}}{1-e^{-\theta x_{(i-1)}^{-\sigma}}}\right]^{\beta-1}} \theta e^{-\theta x_{(i-1)}^{-\sigma}} x_{(i-1)} \ln x_{(x_i)} \left[\left[1-e^{-\theta x_{(i-1)}^{-\sigma}}\right]+e\right.}{\left(1-e^{-\alpha\left[\frac{e^{-\theta x_{(i-1)}^{-\sigma}}}{1-e^{-\theta x_{(i-1)}^{-\sigma}}}\right]^{\beta}}\right)\left[1-e^{-\theta x_{(i-1)}^{-\sigma}}\right]^2}$$

$$\frac{\partial\Omega(x;\sigma,\alpha,\theta,\beta)}{\partial\beta}=\frac{1}{n+1}\sum_{i=1}^{n+1}\frac{1}{\Psi_i}[P_1(x_{(i)};\sigma,\alpha,\theta,\beta)-P_2(x_{(i-1)};\sigma,\alpha,\theta,\beta)] \qquad (36)$$

where

$$M_1(x_{(i)},\sigma,\alpha,\theta,\beta)=\frac{e^{-\alpha\left[\frac{e^{-\theta x_{(i)}^{-\sigma}}}{1-e^{-\theta x_{(i)}^{-\sigma}}}\right]^{\beta}} - \alpha\log\left[\frac{e^{-\theta x_{(i)}^{-\sigma}}}{1-e^{-\theta x_{(i)}^{-\sigma}}}\right]}{1-e^{-\alpha\left[\frac{e^{-\theta x_{(i)}^{-\sigma}}}{1-e^{-\theta x_{(i)}^{-\sigma}}}\right]^{\beta}}}$$

and

$$M_2(x_{(i-1)},\sigma,\alpha,\theta,\beta)=\frac{e^{-\alpha\left[\frac{e^{-\theta x_{(i-1)}^{-\sigma}}}{1-e^{-\theta x_{(i-1)}^{-\sigma}}}\right]^{\beta}}-\alpha\log\left[\frac{e^{-\theta x_{(i-1)}^{-\sigma}}}{1-e^{-\theta x_{(i-1)}^{-\sigma}}}\right]}{1-e^{-\alpha\left[\frac{e^{-\theta x_{(i-1)}^{-\sigma}}}{1-e^{-\theta x_{(i-1)}^{-\sigma}}}\right]^{\beta}}}$$

$$\frac{\partial\Omega(x;\sigma,\alpha,\theta,\beta)}{\partial\alpha}=\frac{1}{n+1}\sum_{i=1}^{n+1}\frac{1}{\Psi_i}[Q_1(x_{(i)};\sigma,\alpha,\theta,\beta)-Q_2(x_{(i-1)};\sigma,\alpha,\theta,\beta)] \qquad (37)$$

The MPS are obtained by setting equations above to zero and solving these equations simultaneously. Thus, these cannot be solved analytically, necessitating the use of analytical tools to solve them numerically.

6. Simulation and Applications

6.1. *Monte-Carlo Simulation*

This section presents Monte-Carlo simulation study to investigate the effect of sample size on the MPS of the parameters of the WGuTII distribution and further to assess the stability of these parameters. Different sample sizes (20, 50, 100, 250, 500, and 1000) were drawn from the WGuTII distribution with parameters $\beta = 1.0$, $\alpha = 1.0$, $\theta = 2.5$, $\sigma = 1.0$ using (17) where each sample was replicated 10000 times. Using the simulated random variables we estimate the parameters of the WGuTII distribution through the methods of MPS and the procedure was repeated 10000 times for each sample size. The corresponding bias and root mean square errors (rmse) of each of the parameter estimates are tabulated in Table 1.

6.2. *Applications to real-life data sets*

In this section we would fit the WGuTII distribution to two real-life data sets to demonstrate its applicability and flexibility. The goodness of fit of WGuTII distribution would be compared three models comprising the baseline distribution, namely, Exponentiated GuTII (EGuTII) distribution, GuTII distribution and Weibull distribution. The model comparison would be based on the minimized log-likelihood estimate and the following information statistics: Akaike information criterion (AIC) and Bayesian information criterion (BIC). The model with the smallest minimized log-likelihood and information statistics value is the best. The first data set consists of 108 observations, representing the COVID-19 mortality rates in Mexico from 4 March to 20 July 2020. This dataset has been used by Almongy et al. [2] and also used by Arif et al. [3]. The data set is provided as follows: 8.826, 6.105, 9.391, 14.962, 10.383, 7.267, 13.220, 16.498, 11.665, 6.015, 10.855, 6.122, 6.656, 3.440, 5.854, 10.685, 10.035, 5.242, 4.344, 5.143, 7.630, 14.604, 7.903, 6.370, 3.537, 6.327, 4.730, 3.215, 9.284, 12.878, 8.813, 10.043, 7.260, 5.985, 6.412, 3.395, 4.424, 9.935, 7.840, 9.550, 3.499, 3.751, 6.968, 3.286, 10.158, 8.108, 6.697, 7.151, 6.560, 2.077, 3.778, 2.988, 3.336, 6.814, 8.325, 7.854,

Table 1. Monte-Carlo simulation results of the parameter estimates using MPS together with their Bias and RMSE of the WGuTII distribution.

n	Parameter	Estimated Values	Bias	RMSE
20	β	1.0698	0.0698	0.3917
	α	1.1572	0.1572	0.3959
	θ	2.5318	0.0318	0.7081
	σ	1.9501	0.9501	0.2976
50	β	1.0575	0.0575	0.2860
	α	1.1584	0.1584	0.3242
	θ	2.5275	0.0275	0.5930
	σ	1.4109	0.4109	0.2345
100	β	1.0698	0.0698	0.3917
	α	1.0423	0.0423	0.2230
	θ	2.5342	0.0342	0.4487
	σ	1.2469	0.2469	0.1781
250	β	1.0252	0.0252	0.1342
	α	1.1138	0.1138	0.1940
	θ	2.5456	0.0456	0.3255
	σ	1.1554	0.1554	0.1176
500	β	1.0129	0.0129	0.0924
	α	1.0973	0.0973	0.1686
	θ	2.5499	0.0499	0.2361
	σ	1.1130	0.1130	0.0846
1000	β	1.0059	0.0059	0.0616
	α	1.0720	0.0720	0.1279
	θ	2.5433	0.0433	0.1704
	σ	1.0101	0.0101	0.0566

8.551, 3.228, 7.486, 6.625, 6.140, 4.909, 4.661, 5.392, 12.042, 8.696, 1.815, 3.327, 5.406, 6.182, 1.041, 1.800, 4.949, 4.089, 3.359, 2.070, 3.298, 5.317, 5.442, 4.557, 4.292, 2.500, 6.535, 4.648, 4.697, 5.459, 4.120, 3.922, 3.219, 1.402, 2.438, 3.257, 3.632, 3.233, 3.027, 2.352, 1.205, 3.218, 2.926, 2.601, 2.065, 3.029, 2.058, 2.326, 2.506, 1.923. The second data set represents the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli, observed and reported by Bjerkedal [4]. The data are given as: 0.1, 0.33, 0.44, 0.56, 0.59, 0.72, 0.74, 0.77, 0.92, 0.93, 0.96, 1, 1, 1.02, 1.05, 1.07, 1.07, 1.08, 1.08, 1.08, 1.09, 1.12, 1.13, 1.15, 1.16, 1.2, 1.21, 1.22, 1.22, 1.24, 1.3, 1.34, 1.36, 1.39, 1.44, 1.46, 1.53, 1.59, 1.6, 1.63, 1.63, 1.68, 1.71, 1.72, 1.76, 1.83, 1.95, 1.96, 1.97, 2.02, 2.13, 2.15, 2.16, 2.22, 2.3, 2.31, 2.4, 2.45, 2.51, 2.53, 2.54, 2.54, 2.78, 2.93, 3.27, 3.42, 3.47, 3.61, 4.02, 4.32, 4.58, 5.55.

Table 2. The models' MPSs and performance requirements based on data set 1

Models	$\hat{\beta}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\sigma}$	ll	AIC	BIC
WGuTII	1.537	3.833	2.365	0.723	-261.173	530.346	541.000
EGuTII	-	9.547	121.623	0.374	-263.728	533.638	541.628
GuTII	-	8.207	-	1.617	-273.819	551.638	556.965
W	-	0.033	-	1.849	-265.451	535.838	541.165

Table 3. The models' MPSs and performance requirements based on data set 2

Models	$\hat{\beta}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\sigma}$	ll	AIC	BIC
WGuTII	4.005	0.948	1.880	0.296	-95.338	198.672	207.779
EGuTII	-	8.362	1280.776	0.237	-97.902	201.804	208.634
GuTII	-	1.048	-	1.130	-118.351	240.702	245.255
W	-	0.301	-	1.721	-99.857	203.714	208.267

7. Discussion of Results

The configuration of the novel model can exhibit either a unimodal shape or monotonic decreasing and increasing shapes. Symmetry and considerable tail variability are observable in Figure 1. The hazard rate function may manifest as unimodal, bathtub-shaped, or inverted bathtub-shaped, contingent upon the values of the shape parameters. The appealing shape characteristics of the new model suggest that the Weibull-Gumbel type II (WGuTII) distribution is well-suited for modeling datasets with non-monotonic hazard rate behaviors frequently encountered in real-life scenarios. The Monte Carlo simulation results in Tables 1 reveal that, as the sample size increases, the parameters of the WGuTII distribution approach the true values, with a simultaneous decrease in bias and root mean square error (RMSE). The diminishing bias and RMSE with increasing sample size indicate improved accuracy in parameter estimation. The outcomes of the model fittings, as presented in Tables 1, 2 and 3 demonstrate that the WGuTII distribution exhibits the best fit for the considered datasets. This conclusion is drawn based on its minimal log-likelihood and information statistics values. Figures 5 depict the empirical and theoretical probability density function (pdf), cumulative density function (cdf), quantile-quantile (Q-Q), and probability-probability (P-P) plots for the estimated distribution of dataset 1, while Figures 6 display the corresponding plots for dataset 2. The close alignment between

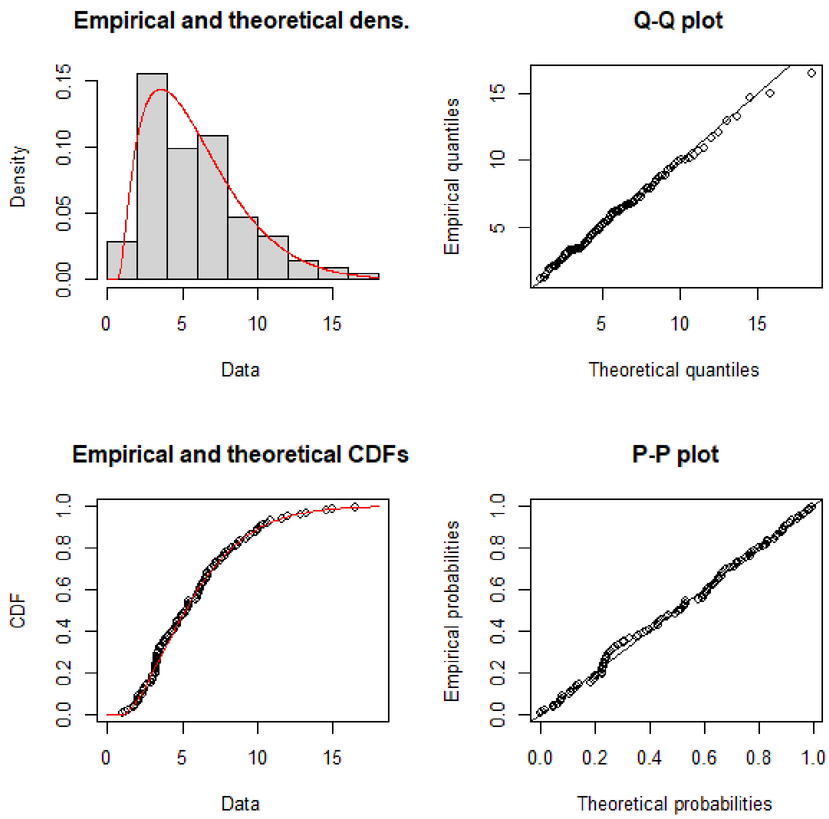

Figure 5. Fitted plots of pdf, cdf, Q-Q and P-P plots for data set 1

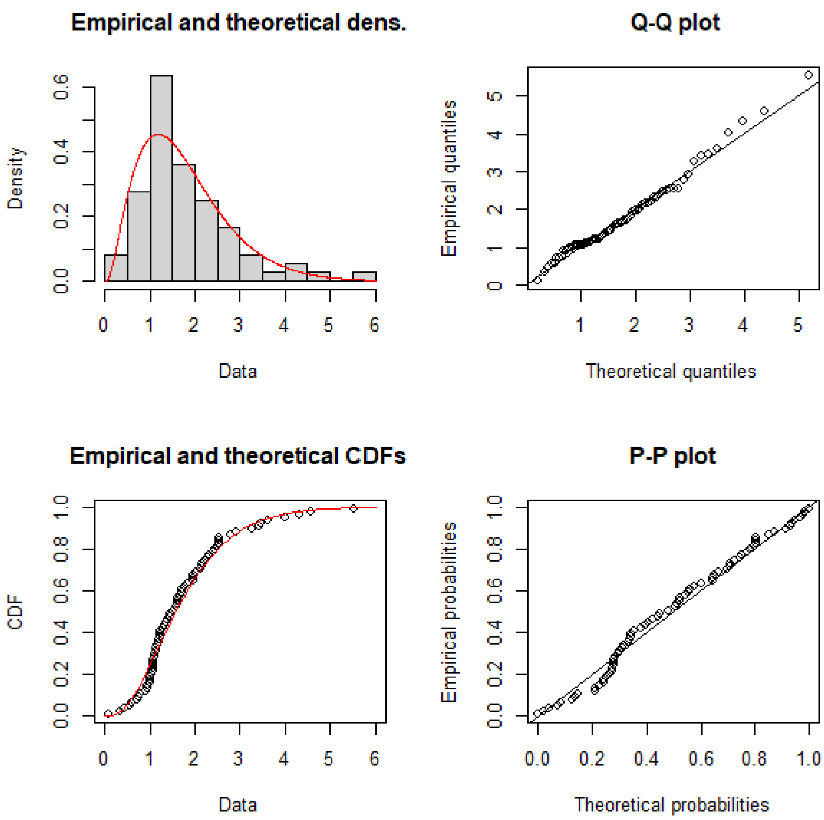

Figure 6. Fitted plots of pdf, cdf, Q-Q and P-P plots for data set 2

the WGuTII distribution and the empirical data in Figures 5 and 6 indicates the suitability of the model for accurately representing the datasets.

8. Conclusion

This paper presents the Weibull-Gumbel type II distribution, a novel lifetime distribution that serves as a generalization of the standard Gumbel type II distribution. The paper provides explicit mathematical expressions for key statistical properties, including the probability density function, cumulative density function, rth moment, moment generating function, survival function, hazard function, quantile function, and both the minimum and maximum order statistics of this new distribution. The study employs two parameter estimation methods, namely maximum likelihood estimation and maximum product of spacing estimation, to determine the values of the unknown parameters. To showcase the adaptability and versatility of the newly introduced lifetime distribution, the authors utilize two real-life datasets. The results of the analysis indicate that, when compared to other related distributions, the Weibull-Gumbel type II distribution offers the most optimal fit. In conclusion, the paper recommends the adoption of the Weibull-Gumbel type II distribution as a robust model for addressing complex datasets. The authors express optimism regarding its potential for significant applications in the future.

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